PHYS 301
Electricity and Magnetism

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Today!

- Magnetic fields
  - Vector potential (briefly!)
  - Magnetic boundary conditions
  - Faraday's Law

Maxwell's Equations

Where do we stand now?

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_o \qquad \text{Gauss' Law } (\vec{E})$$

$$\vec{\nabla}\times\vec{E} = 0 \qquad \text{(Faraday's Law)}$$

ELECTROSTATICS

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \text{Gauss' Law } (\vec{B})$$

$$\vec{\nabla}\times\vec{B} = \mu_o\vec{J} \qquad \text{Ampere's Law}$$

↑ (-Maxwell)

MAGNETOSTATICS

with $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

We're not quite finished!

- electric fields in matter – PHYS 425
- magnetic fields in matter – PHYS 425
- time dependent fields

Two Theorems:

For Curl-less (irrotational) fields:

$i)$ $\vec{\nabla}\times\vec{F} = 0$

$ii)$ $\int_a^b \vec{F}\cdot d\vec{l}$ is path independent

$iii)$ $\oint \vec{F}\cdot d\vec{l} = 0 \;\forall$ closed loops

$iv)$ $\vec{F} = \vec{\nabla} V$

EQUIVALENT!

For Divergence-less (solenoidal) fields:

EQUIVALENT!

$i)$ $\vec{\nabla}\cdot\vec{F} = 0 \;\forall$ space

$ii)$ $\int \vec{F}\cdot d\vec{a}$ is surface independent

$iii)$ $\oint \vec{F}\cdot d\vec{a} = 0 \;\forall$ closed surfaces

$iv)$ $\vec{F} = \vec{\nabla}\times\vec{A}$

Magnetic Vector Potential

Recall that since $\vec{\nabla} \times \vec{E} = 0$ holds for electrostatics, then we can write

$$\vec{E} = -\vec{\nabla} V$$

{ which gave us Laplace's & Poisson's equations from $\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_o$ }

In an analogous way, $\vec{\nabla} \cdot \vec{B} = 0$

allows us to write $\vec{B} = \vec{\nabla} \times \vec{A}$

$\{\ldots \text{ since } \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{V}\right) = 0 \quad \forall \vec{V}\}$

$\vec{A} \equiv$ (magnetic) vector potential

Magnetic Vector Potential

Q: Why doesn't it work to just use a scalar potential, like in electrostatics

***i.e.*, why not just use** $\vec{B} = -\vec{\nabla} U$ **?**

Now, Ampere's Law gives

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_o \vec{J}$$

$$\text{but } \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla}^2 \vec{A} = \mu_o \vec{J}$$

Magnetic Vector Potential

- Note that $\vec{A}$ can be defined within any additive quantity whose curl is zero.

(analogous to electrostatic potential!)

But ....

Turns out - there is a constraint on $\vec{A}$ we impose:

$$\vec{\nabla}\cdot\vec{A}=0$$

simplicity!
[Coulomb Gauge]

So, we are left with

$$\vec{\nabla}\times\vec{B}=\underbrace{-\vec{\nabla}^2\vec{A}=\mu_o\vec{J}}$$

three Poisson's equations!

We've solved this before:

$$\vec{\nabla}^2 V=-\rho/\varepsilon_o \quad \text{gave} \quad V(\vec{r})=\frac{1}{4\pi\varepsilon_o}\int_{vol}\frac{\rho}{\mathscr{r}}d\tau'$$

Magnetic Vector Potential

So, we can write:

$$\boxed{\vec{A}(\vec{r})=\frac{\mu_o}{4\pi}\int_{vol}\frac{\vec{J}}{\mathscr{r}}d\tau'}$$

This can be <u>explicitly</u> derived from the Law of Biot-Savart <u>without</u> the assumption that

$$\vec{\nabla}\cdot\vec{A}=0$$

With a wee bit of work one can show that

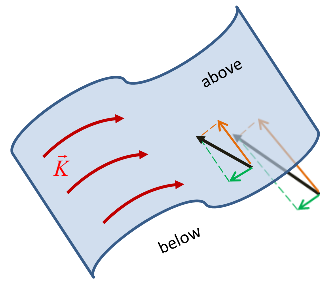
$$\vec{\nabla}\cdot\vec{A}=\frac{\mu_o}{4\pi}\int_{vol}\vec{\nabla}\cdot\left(\frac{\vec{J}}{\mathscr{r}}\right)d\tau'=0.$$

Magnetostatic Boundary Conditions

★ The normal component of $\vec{B}$ is continuous across any boundary

$$\vec{B}_{above} \cdot \hat{n} = \vec{B}_{below} \cdot \hat{n}$$

where $\hat{n} =$ unit vector normal to surface



★ The parallel (tangential) component of $\vec{B}$ that is also perpendicular to the current is discontinuous by an amount $\mu_o K$

$$(\vec{B}_{above} - \vec{B}_{below}) \cdot \hat{\tau} = \mu_o K$$

where $\hat{\tau} =$ unit vector tangent to surface

and $\vec{K} =$ surface current

★ The vector potential is also continuous:

$$\vec{A}_{above} = \vec{A}_{below}$$

so, putting it all together ... $\vec{B}_{above} - \vec{B}_{below} = \mu_o (\vec{K} \times \hat{n})$

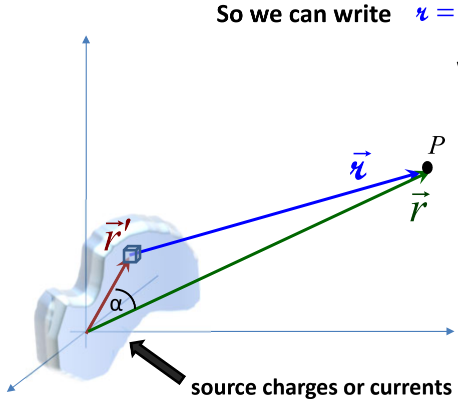
Multipole Expansions of the Potentials

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

$$\mathscr{r}^2 = r^2 + (r')^2 - 2\,r\,r'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

So we can write $\mathscr{r} = r\sqrt{1+\varepsilon}$

where $\varepsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$



For r >> r′ **we can do a binomial expansion on the inverse of this:**

$$\frac{1}{\mathscr{r}} = \frac{1}{r}[1+\varepsilon]^{-1/2}$$

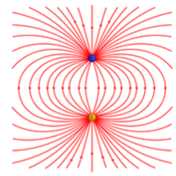
$$= \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \ldots\right)$$

Multipole Expansions of the Potentials

For electrostatic potentials:

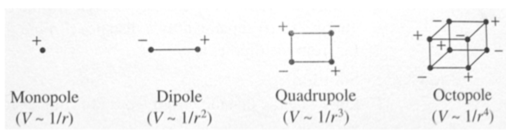
$$V(\vec{r}) = V_{mono}(\vec{r}) + V_{dipole}(\vec{r}) + V_{quad}(\vec{r}) + V_{octa}(\vec{r}) + \ldots$$

where $V_{mono}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r}$ **and** $V_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{\vec{p}\cdot\vec{r}}{r^2}$



from which one can show that

$$\vec{E}_{dipole}(\vec{r}) = \frac{p}{4\pi\varepsilon_o r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) = \frac{1}{4\pi\varepsilon_o}\frac{1}{r^3}[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}]$$



with the electric dipole moment

$$\vec{p} = \int \vec{r}'\rho(\vec{r}')d\tau' = q\,\vec{d}$$

Multipole Expansions of the Potentials

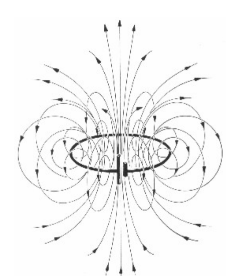
For magnetostatic potentials:

$$\vec{A}(\vec{r}) = \cancel{\vec{A}_{mono}(\vec{r})} + \vec{A}_{dipole}(\vec{r}) + \vec{A}_{quad}(\vec{r}) + \vec{A}_{octa}(\vec{r}) + \ldots$$

where $\vec{A}_{mono}(\vec{r}) = 0$ **and** $\vec{A}_{dipole}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{\vec{m}\times\vec{r}}{r^2}$

with the magnetic dipole moment

$$\vec{m} = I\int d\vec{A} = I\,\vec{A}$$



from which one can show that

$$\vec{B}_{dipole}(\vec{r}) = \frac{\mu_o m}{4\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) = \frac{\mu_o}{4\pi}\frac{1}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}]$$

Faraday's Law

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Faraday's Law

Faraday's Law - some preliminaries

- Physically, what is the meaning of these integrals?

$$\int \vec{B} \cdot d\vec{a} \quad \text{vs.} \quad \oint \vec{B} \cdot d\vec{a}$$

- The flux of the magnetic field through the area of ingegration

$$\text{MAGNETIC FLUX} = \Phi_B = \int \vec{B} \cdot d\vec{a}$$

Faraday's Law - some preliminaries

- THE ELECTROMOTIVE FORCE, $\mathcal{E}$:
  - IT TAKES A FORCE TO MOVE CHARGES:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

(for small speeds, $q\vec{E}$ dominates:

$$\frac{1}{4\pi\varepsilon_o} \sim 9\times10^9\, Nm^2/C^2$$

$$\frac{\mu_o}{4\pi} \sim 1\times10^{-7}\, N/A^2 \quad)$$

  - SO, SINCE MOVING CHARGES CONSTITUTE A CURRENT DENSITY, $\vec{J}$, ONE CAN *EMPIRICALLY* RELATE THIS TO THE DRIVING FORCE:

(Ohm's law) $$\vec{J} = \sigma\left(\frac{\vec{F}}{q}\right) = \sigma\vec{E}$$

conductivity ($\sigma = 1/\rho,\ \rho =$ resistivity)

Faraday's Law - some preliminaries

- THE ELECTROMOTIVE "FORCE" IS DEFINED AS

$$\mathcal{E} = \oint\left(\frac{\vec{F}}{q}\right)\cdot d\vec{\ell} = \oint \vec{E}\cdot d\vec{\ell}$$

FOR A CIRCUIT.

THE *CLOSED LOOP INTEGRAL* IS OVER THE CLOSED CIRCUIT.

- FOR A CIRCUIT IN A MAGNETIC FIELD, A MOTIONAL EMF IS PRODUCED WHEN THERE IS A CHANGE IN THE MAGNETIC FLUX THROUGH THE CIRCUIT:

$$\mathcal{E} = -\frac{d\Phi_E}{dt}$$ "*FLUX RULE*"

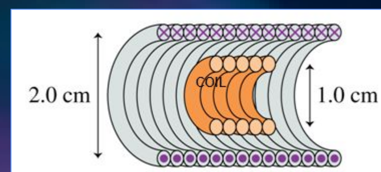
Faraday's Law of Induction

- Suppose you have a long solenoid (d_p = 2.0 cm) with a current flowing through it equal to $i_p(t) = i_o \sin(\omega t)$.
- What flux passes through the secondary coil (N_s = 5) if it's diameter is ½ that of the solenoid (d_s = 1.0 cm) ?
- What if the coil diameter is twice that of the solenoid (d_s = 4.0 cm)?

long solenoid:

$$B_{outside} = 0$$

$$B_{inside} = \mu_o n I$$



$$\mathcal{E} \equiv \text{EMF} = \left| -\frac{d\Phi_B}{dt} \right| = \left| -\frac{d}{dt} \int \vec{B} \cdot d\vec{A} \right|$$

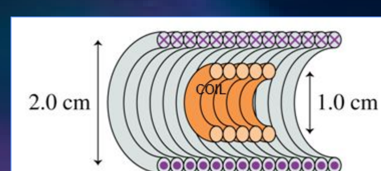
Faraday's Law of Induction

- $i_p(t) = i_o \sin(\omega t)$.
- What EMF is generated in the secondary coil in both cases?
- If the coil has a resistance R_s, what current, $i_s(t)$, flows through it in each case?

long solenoid:

$$B_{outside} = 0$$

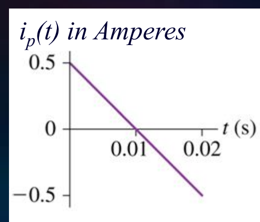
$$B_{inside} = \mu_o n I$$



$$\mathcal{E} \equiv \text{EMF} = \left| -\frac{d\Phi_B}{dt} \right| = \left| -\frac{d}{dt} \int \vec{B} \cdot d\vec{A} \right|$$

Faraday's Law of Induction

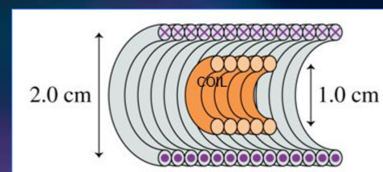
- What if:



long solenoid:

$$B_{outside} = 0$$

$$B_{inside} = \mu_o n I$$



- Now what EMF is generated in the secondary coil in both cases?

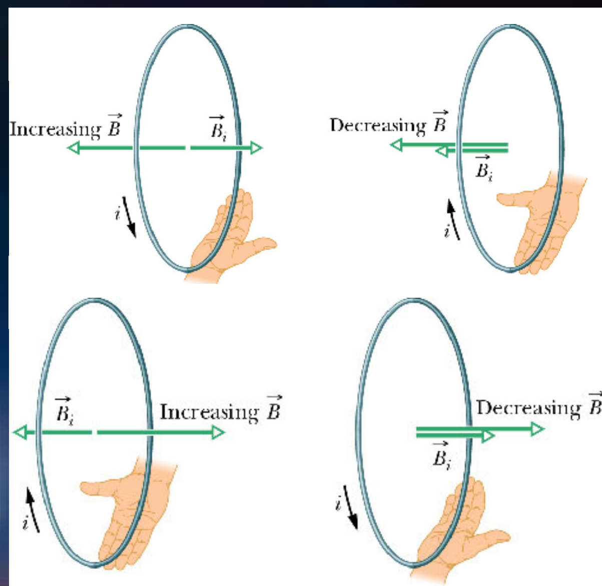
$$\mathcal{E} \equiv \text{EMF} = \left| -\frac{d\Phi_B}{dt} \right| = \left| -\frac{d}{dt} \int \vec{B} \cdot d\vec{A} \right|$$

Lenz's Law

Induced currents oppose the change in flux that created them.

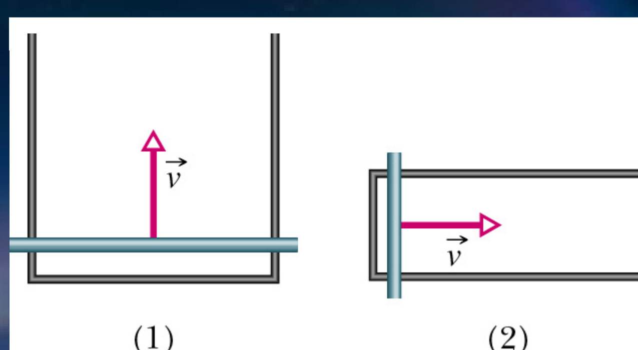
- Helps to determine the direction of the induced current in a loop!
- The induced current will produce its own magnetic field. The direction of this induced field will be such that it opposes the original change in flux.

Lenz's law:



You try!

- For a magnetic field is into the page. What are the directions of the induced currents?



Faraday's Law of Induction

Two versions!

$$EMF = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

* An EMF is induced in any conductor that experiences a change in flux!

Or, more generally,

$$\oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

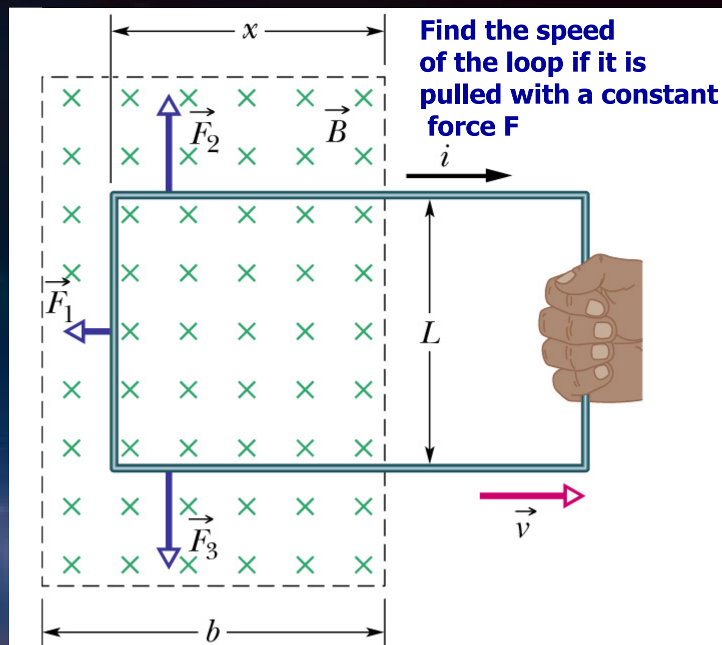
* An electric field is induced in any region of space that experiences a change in flux!

Lentz's Law applies to both versions.

Eddy currents

- If any conductor experiences a change in magnetic flux, a circular electric field will be induced in it.
- This circular electric field will exert forces on any free charges in the conductor, producing circular currents called Eddy currents.

Example:



Faraday's Law

- So we have:

$$\oint \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A} = -\int \frac{d\vec{B}}{dt}\cdot d\vec{A}$$

$$\int (\vec{\nabla}\times\vec{E})\cdot d\vec{A} = -\int \frac{d\vec{B}}{dt}\cdot d\vec{A}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Faraday's Law

Faraday's Law

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Faraday's Law

Faraday's Law

QUASISTATIC APPROXIMATION:

Changing $\vec{B}$'s produce $\vec{E}$'s, but often the formalism of magnetostatics is used to (approximately) describe the changing $\vec{B}$'s!

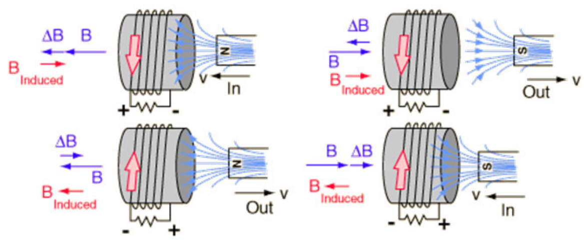
[~ okay, if $\vec{B}$ doesn't change "too rapidly!"]

e.g., fast enough to produce EM waves!

Faraday's Law

dealing with the minus sign …

- **Lenz's Law: If a current flows in a circuit in a magnetic field, it will flow in such a direction that the magnetic field IT produces tends to counteract the change in flux that produced the EMF!**



Faraday's Law

➢ **Things to keep in mind …**

Motional EMFs - magnetic force sets up the EMF, but does not to work!

(it can't – why?!)

whatever is moving the wires (loop – this force is transmitted to charges via structure of the wire.

Magnetic induction – magnet moves (or field changes) but loop is stationary; ∴ charges are not moving so no magnetic force is involved!

Force is due to an electric field produced by the changing magnetic field:

$$\vec{\nabla} \times \vec{E} = -\partial \vec{B} / \partial t$$

❖ Fascinating that the two phenomena can be described in the same way. As Einstein showed, it's the *relative motion* that is important.

Maxwell's Equations

So far, we have . . .

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_o \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_o\vec{J}$$

★ Problem: $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = 0 \quad \forall$ vectors $\vec{V}$

Consider Faraday's law . . .

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) \quad = \quad -\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) \quad = \quad 0$$

Okay for both terms!